Finals Review

Sunday, December 8, 2024 1:27 PM

Spr 2019

Math 241 Spring 2019 Final Exam

- Be sure your name, section number and problem number are on each answer sheet and that you have copied and signed the honor pledge on the first answer sheet.
- Follow the instructions as to which problem goes on which answer sheet. You may use the back of the answer sheets but if you do so, please write "See Back" or something similar on the bottom of the front so we know!
- No calculators or formula sheets are permitted.
- For problems with multiple parts, whether the parts are related or not, be sure to go on to subsequent parts even if there is some part you cannot do.
- Simplification of answers is not necessary. Please leave answers such as $5\sqrt{2}$ or 3π in terms of radicals and π and *do not convert to decimals.*

Please put problem 1 on answer sheet 1

1. Parts (a) and (b) are independent.

   (a) Find parametric equations for the line L containing the points $(-2,0,1)$ and $(4,-2,-3)$. [10 pts]

   (b) Do the planes $\mathcal{P}_0 : 2x - y + 3z = -2$, and $\mathcal{P}_1 : -2x - 3y + z = 6$ intersect? If so, find symmetric equations for the line of intersection. If not, explain why not. [10 pts]

Please put problem 2 on answer sheet 2

2. Parts (a) and (b) are independent.

   (a) Find an equation for the plane containing the points $P = (1,-3,1)$, $Q = (2,2,0)$, and $R = (-4,-1,1)$. [10 pts]

   (b) Find the distance between the point $S = (-2,3,1)$ and the plane $-4x + y - 2z = 0$. [10 pts]

Please put problem 3 on answer sheet 3

3. Parts (a) and (b) are independent.

   (a) Compute the length of the curve C_1 with parametrization: [10 pts]

   $$\mathbf{r}(t) = \tfrac{1}{3}(1+t)^{\frac{3}{2}}\,\mathbf{i} + \tfrac{1}{3}(1-t)^{\frac{3}{2}}\,\mathbf{j} + t\sqrt{3}\,\mathbf{k} \text{ for } -\tfrac{1}{4} \le t \le \tfrac{1}{2}$$

   (b) Find all points (if any) where the curve C_2 with the following parametrization meets the sphere of radius 3 centered at the origin: [10 pts]

   $$\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t+1}\,\mathbf{j} + t\,\mathbf{k} \text{ for } t \ge 0$$

Please put problem 4 on answer sheet 4

4. Consider the curve parameterized by $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$.

   (a) Find the tangent vector $\mathbf{T}(t)$. [12 pts]

   (b) Find the normal vector $\mathbf{N}(t)$. [8 pts]

Turn Over!

Please put problem 5 on answer sheet 5

5. Let $f(x,y,z) = z^2 - 8\sqrt{x^2 - 3y}$.

   (a) Find grad$f$. [5 pts]

   (b) Find an equation of the plane tangent to the level surface for f at $(5,3,2)$. [5 pts]

   (c) Find $D_{\mathbf{u}}f$ at $(5,3,2)$ where $\mathbf{u}$ is pointing in the direction $1\,\mathbf{i} + 2\,\mathbf{j} - 3\,\mathbf{k}$. [5 pts]

   (d) Find the smallest value of $D_{\mathbf{u}}f$ at $(5,3,2)$. [5 pts]

Please put problem 6 on answer sheet 6

6. Let $f(x,y) = x^4 + y^2$.

   (a) Use Lagrange Multipliers to find the maximum and minimum of $f(x,y)$ subject to the constraint $x^2 + y^2 = 1$. [15 pts]

   (b) Find the maximum and minimum of $f(x,y)$ subject to the constraint $x^2 + y^2 \le 1$. [5 pts]

Please put problem 7 on answer sheet 7

7. Use the change of variables $u = y - x$ and $v = y + x$ to evaluate the double integral: [20 pts]

   $$\iint_R (y-x)\sin\left((y+x)^3\right)\,dA$$

   where R is the triangle with vertices $(0,0)$, $(2,2)$ and $(0,4)$.

Please put problem 8 on answer sheet 8

8. Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the paraboloid $2z = x^2 + y^2$. [20 pts]

Please put problem 9 on answer sheet 9

9. Parts (a) and (b) are independent.

   (a) Let Σ be the portion of the cylinder $x^2 + y^2 = 9$ between $z = 1$ and $z = 8$. If the mass density at (x,y,z) is given by $f(x,y,z) = x^2 z$ write down an iterated double integral for the mass of Σ.
   Do not evaluate the integral! [12 pts]

   (b) Let C be the triangle with vertices $(0,4)$, $(2,0)$ and $(2,4)$ with clockwise orientation. Use Green's Theorem to evaluate: [8 pts]

   $$\int_C 4y\,dx + 9x\,dy$$

Please put problem 10 on answer sheet 10

10. Let Σ be the portion of the plane $x + 2y + z = 10$ in the first octant. Let C be the boundary of Σ with counterclockwise orientation when viewed from above. Use Stokes' Theorem to rewrite the integral $\int_C 3xy\,dx + z^2\,dy + xy\,dz$ as a surface integral and then proceed until you have an iterated double integral. **Do not evaluate the integral!** [20 pts]

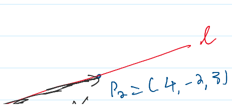
Welcome to the End of the Exam

Please put problem 1 on answer sheet 1

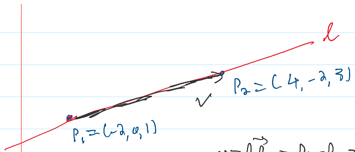
1. Parts (a) and (b) are independent.

   (a) Find parametric equations for the line L containing the points $(-2,0,1)$ and $(4,-2,-3)$. [10 pts]

   (b) Do the planes $\mathcal{P}_0 : 2x - y + 3z = -2$, and $\mathcal{P}_1 : -2x - 3y + z = 6$ intersect? If so, find symmetric equations for the line of intersection. If not, explain why not. [10 pts]



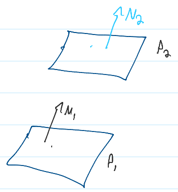
Recall to find a line we need
- a point P ✓
- a parallel vector v



Recall to find a line we need $\begin{cases} \bullet \text{ a point } P \checkmark \\ \bullet \text{ a parallel vector } v \end{cases}$

$v = \overrightarrow{P_1P_2} = P_2 - P_1 = (6, -2, 2)$

$$\ell: \begin{matrix} x = \\ y = \\ z = \end{matrix} \underbrace{\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}}_{P_1} + \underbrace{\begin{pmatrix} 6 \\ -2 \\ 2 \end{pmatrix}}_{v} t$$



Rmk: $P_0 \parallel P_1$ if $N_0 \parallel N_1$

[6] $P_0: 2x - y + 3z = -2$

$N_0 = (2, -1, 3)$

$P_1: -2x - 3y + z = 6$

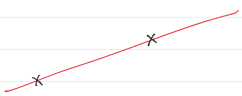
$N_1 = (-2, -3, 1)$

Notice: $(2, -1, 3) \neq c\,(-2, -3, 1)$ so they are not parallel

so the planes intersect,

and the intersection is a line

• To find the line, we will just find two points on the line



So I have to find two points of common intersection between the 2 planes

$P_0: 2x - y + 3z = -2$

$P_1: -2x - 3y + z = 6$

• First Point set $z = 0$: $\begin{cases} 2x - y = -2 \\ -2x - 3y = 6 \end{cases} \to \oplus$ $-4y = 4 \Rightarrow y = -1 \Rightarrow 2x + 1 = -2$

$\Rightarrow 2x = -3$

$\Rightarrow x = -\frac{3}{2}$

$\boxed{Q_1 = (-\frac{3}{2}, -1, 0)}$

• Second Point: $y = 0$: $\begin{cases} 2x + 3z = -2 \\ -2x + z = 6 \end{cases} \to \oplus$ $4z = 4 \Rightarrow z = 1 \Rightarrow 2x + 3 = -2$

$\Rightarrow 2x = -5$

$\Rightarrow x = -\frac{5}{2}$

$\boxed{Q_2 = (-\frac{5}{2}, 0, 1)}$

$v = \overrightarrow{Q_1Q_2} = (-1, +1, 1)$

So line of intersection: Vector eq: $r(t) = Q_1 + \overrightarrow{Q_1Q_2}\, t$

$$r(t) = \begin{bmatrix} -\frac{3}{2} \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix} t \quad \leadsto$$

Parametric

$x = -\frac{3}{2} - t \quad \Rightarrow t = -x - \frac{3}{2}$

$y = -1 + t \quad t = y + 1$

$z = 0 + t \quad t = z$

$\Rightarrow$ Symmetric $\boxed{-x - \frac{3}{2} = y + 1 = z}$

$x = x_0 + at \leadsto t = \frac{x - x_0}{a}$

$y = y_0 + bt \leadsto t = \frac{y - y_0}{b}$

$z = z_0 + ct \leadsto t = \frac{z - z_0}{c}$

Symm: $\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$

Please put problem 2 on answer sheet 2

2. Parts (a) and (b) are independent.

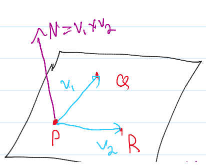
(a) Find an equation for the plane containing the points $P = (1, -3, 1)$, $Q = (2, 2, 0)$, and $R = (-4, -1, 1)$. [10 pts] in the form $ax + by + cz = d$

(b) Find the distance between the point $S = (-2, 3, 1)$ and the plane $-4x + y - 2z = 0$. [10 pts]

(a) To find a plane we need (i) a point P_0

(ii) a normal vector N or two vectors v_1, v_2 parallel to plane

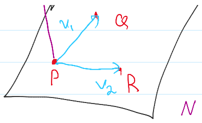
In that case $N = v_1 \times v_2$



$v_1 = \overrightarrow{PQ} = (1, 5, -1)$

$v_2 = \overrightarrow{PR} = (-5, 2, 0)$

$$N = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 5 & -1 \\ -5 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 2 \\ 5 \\ 27 \end{bmatrix}$$



$v_1 = PQ = (1, 5, -1)$

$v_2 = PR = (-5, 2, 0)$

$$N = v_1 \times v_2 = \begin{vmatrix} i & j & k \\ 1 & 5 & -1 \\ -5 & 2 & 0 \end{vmatrix} = \begin{bmatrix} 2 \\ 5 \\ 27 \end{bmatrix}$$

$$E: 2(x-1) + 5(y+3) + 27(z-1) = 0$$

P

(b) $E: -4x + y - 2z = 0 \rightsquigarrow N = (-4, 1, -2)$ normal vector

$P = (0,0,0)$ pt on the plane

$S = (-2, 3, 1)$

$$\text{dist}(S, E) = \frac{|N \cdot SP|}{\|N\|} = \frac{\left| \begin{bmatrix} -4 \\ 1 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -3 \\ -1 \end{bmatrix} \right|}{\|(-4, 1, -2)\|} = \frac{|-8-3+2|}{\sqrt{16+1+4}} = \frac{9}{\sqrt{21}}$$

In general: $E: 2x + 3y + 4z = 4$

To find a point, set $y = z = 0$

Then $2x = 4 \Rightarrow x = 2 \Rightarrow P = (2, 0, 0)$ pt on the plane

$S = (-2, 3, 1)$

$P = (0,0,0)$

$SP = P - S = (2, -3, -1)$

You could also take $PS = S = (-2, 3, 1)$

Please put problem 3 on answer sheet 3

3. Parts (a) and (b) are independent.

   (a) Compute the length of the curve C_1 with parametrization: [10 pts]

   $$\mathbf{r}(t) = \tfrac{1}{3}(1+t)^{\frac{3}{2}}\,\mathbf{i} + \tfrac{1}{3}(1-t)^{\frac{3}{2}}\,\mathbf{j} + t\sqrt{3}\,\mathbf{k} \text{ for } -\tfrac{1}{4} \le t \le \tfrac{1}{2}$$

   (b) Find $\boxed{\text{all points}}$ (if any) where the curve C_2 with the following parametrization meets the sphere of radius 3 centered at the origin: [10 pts]

   $$\mathbf{r}(t) = \sqrt{t}\,\mathbf{i} + \sqrt{t+1}\,\mathbf{j} + t\,\mathbf{k} \text{ for } t \ge 0$$

$$r'(t) = \begin{bmatrix} \frac{1}{2}\sqrt{1+t} \\ -\frac{1}{2}\sqrt{1-t} \\ \sqrt{3} \end{bmatrix}$$

(a) $$L = \int_a^b \left\|\frac{dr}{dt}\right\| dt = \int_{-\frac{1}{4}}^{\frac{1}{2}} \sqrt{\tfrac{1}{4}(1+t) + \tfrac{1}{4}(1-t) + 3}$$

$$= \int_{-\frac{1}{4}}^{\frac{1}{2}} \sqrt{\tfrac{1}{2} + 3} = \sqrt{\tfrac{7}{2}} \cdot \left(\tfrac{1}{2} + \tfrac{1}{4}\right)$$

$$= \sqrt{\tfrac{7}{2}} \cdot \tfrac{3}{4}$$

(b) $$r(t) = \begin{bmatrix} \sqrt{t} \\ \sqrt{t+1} \\ t \end{bmatrix} \begin{matrix} x \\ y \\ z \end{matrix}$$ $\rightsquigarrow t \geq 0$

Sphere: $x^2 + y^2 + z^2 = 9$

$$x^2 + y^2 + z^2 = t + (t+1) + t^2 = 9$$

$$\Rightarrow t^2 + 2t + 1 = 9 \qquad t^2 + 2t - 8 = 0 \qquad t = 2,\ \cancel{-4}$$

$$P = r(2) = \begin{bmatrix} \sqrt{2} \\ \sqrt{3} \\ 2 \end{bmatrix}$$ = point of intersection

Please put problem 4 on answer sheet 4

4. Consider the curve parameterized by $\mathbf{r}(t) = \cos^3 t\,\mathbf{i} + \sin^3 t\,\mathbf{j}$.

   (a) Find the tangent vector $\mathbf{T}(t)$. [12 pts]

   (b) Find the normal vector $\mathbf{N}(t)$. [8 pts]

$$r'(t) = \begin{bmatrix} -3\cos^2(t)\sin(t) \\ 3\sin^2(t)\cos(t) \end{bmatrix} \qquad T(t) = \frac{r'(t)}{\|r'(t)\|}$$

$$\|r'(t)\| = \sqrt{9\cos^4(t)\sin^2(t) + 9\sin^4(t)\cos^2(t)} = \sqrt{9\cos^2(t)\sin^2(t)\cdot\underbrace{(\cos^2(t)+\sin^2(t))}_{1}} = 3\cos(t)\sin(t)$$

Always simplify before moving to the next step

$$T(t) = \frac{1}{3\cos(t)\sin(t)} \cdot \begin{bmatrix} -3\cos^2(t)\sin(t) \\ 3\sin^2(t)\cos(t) \end{bmatrix} = \begin{bmatrix} -\cos(t) \\ \sin(t) \end{bmatrix}$$ = tangent vector

$$N(t) = \frac{T'(t)}{\|T'(t)\|}$$

$$T'(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix} \rightsquigarrow \|T'(t)\| = 1$$

$$\Rightarrow N(t) = \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$$ = normal vector

Tangent planes: (1) $f(x,y,z)=c$ level set then $\nabla f = N$

e.g. $x^2+y^2+z^2=4$ then $N=\nabla f=(2x,2y,2z)$

(2) $z=f(x,y)$ then $g=f(x,y)-z$ and $N=\nabla g$

e.g. $f(x,y)=x^2+3y^2$ $g=x^2+3y^2-z$ and $N=\nabla g=(2x,6y,-1)$

Please put problem 5 on answer sheet 5

5. Let $f(x,y,z)=z^2-8\sqrt{x^2-3y}$.
   - (a) Find grad f. [5 pts]
   - (b) Find an equation of the plane tangent to the level surface for f at $(5,3,2)$. [5 pts]
   - (c) Find $D_{\mathbf{u}}f$ at $(5,3,2)$ where $\mathbf{u}$ is pointing in the direction $1\mathbf{i}+2\mathbf{j}-3\mathbf{k}$. [5 pts]
   - (d) Find the smallest value of $D_{\mathbf{u}}f$ at $(5,3,2)$. [5 pts]

$$\nabla f=\begin{bmatrix} f_x \\ f_y \\ f_z\end{bmatrix}=\begin{bmatrix} -8\cdot\frac{1}{2\sqrt{x^2-3y}}\cdot 2x \\ -8\cdot\frac{1}{2\sqrt{x^2-3y}}\cdot(-3) \\ 2z\end{bmatrix}=\begin{bmatrix}\frac{-8x}{\sqrt{x^2-3y}} \\ \frac{12}{\sqrt{x^2-3y}} \\ 2z\end{bmatrix}$$

(b) $N=\nabla f(5,3,2)=\begin{bmatrix}\frac{-8\cdot 5}{4} \\ \frac{12}{4} \\ 4\end{bmatrix}=(-10,3,4)$ $\rightsquigarrow$ E: $-10(x-5)+3(y-3)+4(z-2)=0$

(c) Th: (1) D_uf maximizes at $u=\nabla f$ and max $=\|\nabla f\|$

(2) D_uf minimizes at $u=-\nabla f$ and min $=-\|\nabla f\|$ $\leftarrow$

(3) $D_uf=0$ when $u\perp\nabla f$

In general $D_uf=\nabla f\cdot\frac{u}{\|u\|}$

$u=(1,2,-3)$ $\rightsquigarrow$ $D_uf=\left(\nabla f\cdot\frac{u}{\|u\|}\right)(5,3,2)$

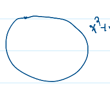
$=\nabla f(5,3,2)\cdot\frac{u}{\|u\|}$

$=(-10,3,4)\cdot\frac{(1,2,-3)}{\sqrt{1+4+9}}=\frac{1}{\sqrt{14}}\cdot(-10+6-12)=\frac{-16}{\sqrt{14}}$

(d) The smallest value is $-\|\nabla f\|=-\|(-10,3,4)\|=-\sqrt{100+9+16}=-\sqrt{125}$

Please put problem 6 on answer sheet 6

6. Let $f(x,y)=x^4+y^2$.
   - (a) Use Lagrange Multipliers to find the maximum and minimum of $f(x,y)$ subject to the constraint $x^2+y^2=1$. [15 pts]
   - (b) Find the maximum and minimum of $f(x,y)$ subject to the constraint $x^2+y^2\le 1$. [5 pts]



$x^2+y^2=1$

$g(x,y)=x^2+y^2 \rightsquigarrow \nabla g=\begin{bmatrix}2x\\2y\end{bmatrix}$

$f(x,y)=x^4+y^2 \rightsquigarrow \nabla f=\begin{bmatrix}4x^3\\2y\end{bmatrix}$

Lagranges $\begin{cases}\nabla f=\lambda\nabla g \\ \text{constraint}\end{cases}$ $\longrightarrow$ $\begin{cases}4x^3=2\lambda x \\ 2y=2\lambda y \\ x^2+y^2=1\end{cases}$ $\rightsquigarrow$ $2y-2\lambda y=0 \Rightarrow 2y(1-\lambda)=0$ $\nearrow \boxed{y=0}$ or $\searrow \boxed{\lambda=1}$

Case 1: $y=0$: $x^2+0^2=1 \Rightarrow x=\pm 1$ $\rightsquigarrow$ $P_1=(1,0)$ $P_2=(-1,0)$

Case 2: $\lambda=1$: $4x^3=2x \Rightarrow 4x^3-2x=0 \Rightarrow 2x(2x^2-1)=0 \Rightarrow x=0$ or $x=\pm\frac{\sqrt{2}}{2}$

- If $x=0 \Rightarrow 0+y^2=1 \Rightarrow y=\pm 1$ $\rightsquigarrow$ $P_3=(0,1)$ $P_4=(0,-1)$
- If $x=+\frac{\sqrt{2}}{2} \Rightarrow y=\pm\frac{\sqrt{2}}{2} \Rightarrow P_5=\left(\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$ $P_6=\left(\frac{\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right)$
- If $x=-\frac{\sqrt{2}}{2} \Rightarrow y=\pm\frac{\sqrt{2}}{2} \Rightarrow P_7=\left(-\frac{\sqrt{2}}{2},\frac{\sqrt{2}}{2}\right)$, $P_8=\left(\frac{-\sqrt{2}}{2},\frac{-\sqrt{2}}{2}\right)$

$f=x^4+y^2$

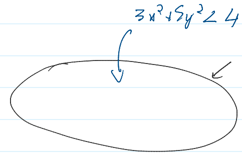
- $f(P_1)=1$
- $f(P_2)=1$
- $f(P_3)=1$
- $f(P_4)=1$
- $f(P_5)=\frac{1}{4}+\frac{1}{2}$
- $f(P_6)=\frac{1}{4}+\frac{1}{2}$
- $f(P_7)=\frac{1}{4}+\frac{1}{2}$
- $f(P_8)=\frac{1}{4}+\frac{1}{2}$

max $=1$

min $=\frac{3}{4}$

Interior $\nabla f=0$

$3x^2+5y^2<4$



$3x^2+5y^2=4$ boundary Lagrange

$3x^2+5y^2\le 4$

(b) I have to look at the interior

Crit pts: $\nabla f=0$ $\leftarrow$

$\nabla f=\begin{bmatrix}4x^3\\2y\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix} \Rightarrow x=y=0 \Rightarrow Q=(0,0)$

$f(Q) = 0$

so abs min $= 0$

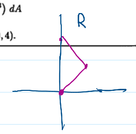
abs max $= 1$

Please put problem 7 on answer sheet 7

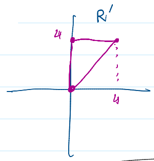
7. Use the change of variables $u = y - x$ and $v = y + x$ to evaluate the double integral: [20 pts]

$$\iint_R (y - x)\sin\left((y + x)^3\right)\, dA$$

where R is the triangle with vertices $(0, 0)$, $(2, 2)$ and $(0, 4)$.



$\rightsquigarrow$



$u = y - x$ $\quad \overset{\oplus}{\Rightarrow}\ 2y = u + v \Rightarrow y = \frac{1}{2}(u+v)$

$v = y + x$ $\quad \ominus\ 2x = v - u \Rightarrow x = \frac{1}{2}(v - u)$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = -\frac{1}{2}$$

$u = y - x$ In terms of y, v:

$v = y + x$

$P_1 = (0,0) \rightsquigarrow P_1' = (0-0, 0+0) = (0,0)$

$P_2 = (2,2) \rightsquigarrow P_2' = (2-2, 2+2) = (0,4)$

$P_3 = (0,4) \rightsquigarrow P_3' = (4-0, 4+0) = (4,4)$

1) Either solve x, y in terms of y, v and find J

2) Or compute $J' = \begin{vmatrix} \frac{\partial y}{\partial x} & \frac{\partial y}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} = -2$

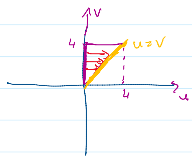
then $\det(J) = \dfrac{1}{\det(J')} = \boxed{-\frac{1}{2}}$

$$\iint_R (y-x)\sin\left((y+x)^3\right) dA$$

$$= \iint_{R'} u \cdot \sin(v^3) \cdot |\det(J)|\, dA$$

$$= -\frac{1}{2} \iint_{R'} u \sin(v^3)\, dA$$

$du\, dv$



! I cannot compute $\int \sin(v^3)\, dv$, so I will move horizontally to integrate over u first.

$$= -\frac{1}{2} \int_{v=0}^{v=4} \int_{u=0}^{u=v} u \sin(v^3)\, du\, dv = -\frac{1}{2} \int_0^4 \frac{v^2}{2} \sin(v^3)\, dv$$

$$= -\frac{1}{4} \int_0^4 v^2 \sin(v^3)\, dv$$

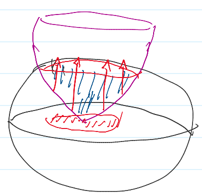
$$= -\frac{1}{4} \cdot \frac{1}{3} \int_0^{4^3} \sin(w)\, dw = \frac{1}{12}\cos(4^3) - \frac{1}{12}$$

$w = v^3$

$dw = 3v^2\, dv$

Please put problem 8 on answer sheet 8

8. Find the volume of the solid region bounded above by the sphere $x^2 + y^2 + z^2 = 8$ and below by the paraboloid $2z = x^2 + y^2$. [20 pts]



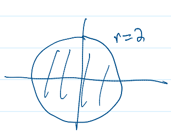
$$\int\int_R \int_{z = \frac{1}{2}(x^2+y^2)}^{z = \text{sphere}} 1$$

$$= \int\int_R \int_{z = \frac{1}{2}r^2}^{z = \sqrt{8 - r^2}} r \quad dz\, dr\, d\theta$$

$2z = x^2 + y^2$, $x^2 + y^2 + z^2 = 8$ $\Rightarrow 2z = 8 - z^2 \Rightarrow z^2 + 2z - 8 = 0 \rightsquigarrow \boxed{z = 2}$

$z = -4$

$z = 2$: $x^2 + y^2 + 4 = 8 \rightarrow \boxed{x^2 + y^2 = 4}$



$$= \int_{\theta=0}^{2\pi} \int_{r=0}^{2} \int_{z = \frac{1}{2}r^2}^{z = \sqrt{8 - r^2}} r\, dz\, dr\, d\theta = \dots = \text{computations}$$

Please put problem 9 on answer sheet 9

9. Parts (a) and (b) are independent.

(a) Let Σ be the portion of the cylinder $x^2 + y^2 = 9$ between $z = 1$ and $z = 8$. If the mass density at (x, y, z) is given by $f(x, y, z) = x^2 z$ write down an iterated double integral for the mass of Σ.
Do not evaluate the integral! [12 pts]

(b) Let C be the triangle with vertices $(0, 4)$, $(2, 0)$ and $(2, 4)$ with clockwise orientation. Use Green's Theorem to evaluate: [8 pts]

$$\int_C 4y\, dx + 9x\, dy$$

$x^2 + y^2 = 9$

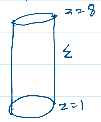
$r = 3$

(b) Let C be the triangle with vertices $(0,4)$, $(2,0)$ and $(2,4)$ with clockwise orientation. Use Green's Theorem to evaluate: [8 pts]

$$\int_C 4y\,dx + 9x\,dy$$

$x^2+y^2=9$

$r=3$



$$\iint_\Sigma \underset{\sim}{f} \cdot dS$$

$$r(\theta,z) = \begin{bmatrix} 3\cos(\theta) \\ 3\sin(\theta) \\ z \end{bmatrix}$$

$$r_\theta \times r_z = \begin{vmatrix} i & j & k \\ -3\sin\theta & 3\cos(\theta) & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{bmatrix} 3\cos(\theta) \\ 3\sin(\theta) \\ 0 \end{bmatrix}$$

$f = x^2 \cdot z$

$$\|r_\theta \times r_z\| = 3$$

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

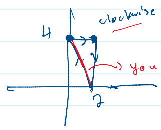
$$\iint_\Sigma f \cdot dS = \iint f(r(\theta,z)) \cdot \|r_\theta \times r_z\|$$

$$= \int_{z=1}^{z=8}\int_{\theta=0}^{\theta=2\pi} \underbrace{9\cos^2(\theta)\cdot z}_{f(r(\theta,z))} \cdot \underbrace{3}_{\|r_\theta \times r_z\|} = 27 \int_1^8\int_0^{2\pi} \cos^2(\theta)\cdot z \; d\theta\, dz$$

$$= 27\int_1^8 \int_0^{2\pi} \frac{1+\cos(2\theta)}{2} \cdot z \; d\theta \; dz$$

$$= 27\int_1^8 \pi \cdot z \; dz = 27\pi \frac{63}{2}$$

(b)



you'd have to find the eqn of this line $y = -2x+4$

$$\int_C F \cdot dr \quad \text{where} \quad F = \begin{bmatrix} 4y \\ 9x \end{bmatrix}$$

clockwise

$$\int_C F\cdot dr = (-) \iint N_x - M_y \; dA$$

$$= -\iint 9-4 \; dA$$

$$= -5 \iint 1 \; dA$$

$$= -5 \cdot (\text{area of triangle})$$

$$= -5 \cdot \frac{\text{base} \cdot \text{height}}{2}$$

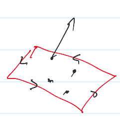
$$= \frac{-5\cdot 2\cdot 4}{2} = -20$$

Please put problem 10 on answer sheet 10

10. Let Σ be the portion of the plane $x+2y+z=10$ in the first octant. Let C be the boundary of Σ with counterclockwise orientation when viewed from above. Use Stokes' Theorem to rewrite the integral $\int_C 3xy\,dx + z^2\,dy + xy\,dz$ as a surface integral and then proceed until you have an iterated double integral. **Do not evaluate the integral!** [20 pts]

$$\int_C F\cdot dr = \pm \iint_\Sigma \text{curl}(F)\cdot n \; dS$$

$$F = \begin{bmatrix} 3xy \\ z^2 \\ xy \end{bmatrix} \rightsquigarrow \text{curl}(F) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3xy & z^2 & xy \end{vmatrix} = \begin{bmatrix} x-2z \\ -y \\ -3x \end{bmatrix}$$

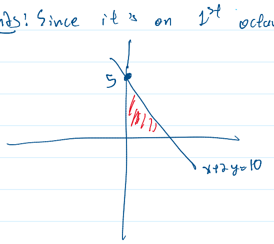


• $x+2y+z=10 \rightsquigarrow r(x,y) = \begin{bmatrix} x \\ y \\ \underbrace{10-x-2y}_{f(x,y)} \end{bmatrix} \rightsquigarrow r_x \times r_y = \begin{bmatrix} -f_x \\ -f_y \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \rightsquigarrow$ orientation

$$\text{So } +\iint_\Sigma \text{curl}(F)\cdot n \; dS = +\iint \text{curl}(F)(r(x,y)) \cdot (r_x \times r_y)\; dA$$

$$= +\iint \begin{bmatrix} x - 2(10-x-2y) \\ -y \\ -3x \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} dA$$

Bounds: Since it's on 1st octant, set $z=0$: $x+2y=10$



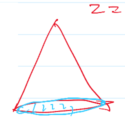
$$+\int_{y=0}^{5}\int_{x=0}^{x=10-2y} \underbrace{x - 20 + 2x + 4y - 2y - 3x}_{\substack{\shortparallel \\ 2x+2y-20}} \; dx\; dy$$

$$\int_C F\cdot dr = \iint \underbrace{\mathrm{curl}(F)}_{G}\cdot n \, ds$$

10. (a) (10 points) Let D be the solid object above the xy-plane and below the cone $z = 4 - \sqrt{x^2+y^2}$. Let Σ be the boundary of D, oriented inwards. (So Σ is a piecewise smooth surface consisting of the cone as well as the bottom disk in the xy-plane.) Evaluate the following integral

$$\int\int_{\Sigma} (2x\boldsymbol{i} + 5x\boldsymbol{j} + 7z\boldsymbol{k}) \cdot \boldsymbol{n} \, dS.$$

$z = 4-\sqrt{x^2+y^2}$

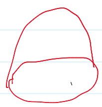


$z=0$:

$x^2+y^2 = 4^2$

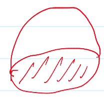
Divergence is used on CLOSED Surfaces

upper hemisphere



Cannot use divergence

upper hemisphere with the disk at the bottom



Divergence

$$\iint_\Sigma F\cdot n\, ds = \underset{\text{inwards}}{\overset{\text{outward}}{\pm}} \iiint_D \mathrm{div}(F)\, dV$$

$$F = \begin{bmatrix} 2x \\ 5x \\ 7z \end{bmatrix} \rightsquigarrow \mathrm{div}(F) = 2+7 = 9$$

$$= -\iiint_D 9 \, dV$$

$z = 4-\sqrt{x^2+y^2}$

$x^2+y^2 = 4^2$

$$= -\iint_R \int_{z=0}^{z=4-\sqrt{r^2}} 9$$

R

$x^2+y^2 \le 4$

$$= -\int_0^{2\pi}\int_0^4\int_0^{4-r} 9r \, dz\, dr\, d\theta$$

= computations.